The monitoring of mean vectors with VCS charts for multivariate processes

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ABSTRACT – In this article, we propose to control the mean vector of multivariate processes by taking larger samples but inspecting fewer quality characteristics at each inspection. For instance, if we decide to work with samples of size n to control bivariate processes, then $2n$ observations are usually collected, half are observations of variable X and the other half are observations of variable Y; alternatively, we might work with samples of size $2n$ if only observations of X (or Y) are collected; in both cases the cardinality of the sample data set is $2n$. If only one of the two quality characteristics, X or Y, is measured at each sampling time, then only one of the two statistics ($\overline{X}$ or $\overline{Y}$) is computed. The variable charting statistic (VCS) chart works as follows: if a $\overline{X}$ point falls in the central region, then the statistic for the next sample changes to $\overline{Y}$ and vice-versa; yet, if a $\overline{X}$ point reaches the warning region, then variable X will be measured again. For the trivariate case, the sequence with which the charting statistic changes is: $\overline{X}$ to $\overline{Y}$, $\overline{Y}$ to $\overline{Z}$, and $\overline{Z}$ to $\overline{X}$. We also applied the VCS strategy to control four quality characteristics (X, Y, Z, and W); the Hotelling T^2 statistic obtained with the X and Y observations and the Hotelling T^2 statistic obtained with the Z and W observations are the two statistics of the VCS chart. In comparison with the standard T^2 chart, the proposed VCS charts are not only simpler to use but also faster to present an alarm.

Keywords – Variable charting statistic, alternate charting statistic, X-bar control chart, T^2 Hotelling, multivariate processes

1. Introduction

The idea of varying the control chart's parameters is not recent, but remains alive. In their pioneer article, Reynolds et al[1] worked with variable sampling intervals; subsequently, Costa[2] worked with variable sample sizes and, five years later, Costa[3] studied the Shewhart charts with three design parameters varying between two levels: the sampling intervals, the sample sizes and the width of the control limits. Recent articles, where the properties of the control charts with variable parameters are investigated, include Abolmohammadi et al[4], Chong et al[5], Katebi and Moghadam[6], Sabahno et al[7], Cheng and Wang[8], Coelho et al[9], Pourtaheri[10], Zhang et al[11], Zhou[12], Yue and Liu[13].

More recently, Costa and Faria Neto[14] considered the strategy of varying the charting statistic. They proposed to control the covariance matrix of bivariate processes with *S* charts. That means, at each sampling point, only one of the two quality characteristics, *X* or *Y*, is measured and only one of the two statistics (S_x or S_y) is computed. When the control chart with variable charting statistic is used, the type (S_x or S_y) and the position of the current sample point on the chart define the statistic for the next sampling point. If the current point is the standard deviation of the *X* values and it is in the central region, then the statistic for the next sample changes to the standard deviation of the *Y* values. However, if the current point is located in the warning region, the variable remains the same (in this case, the standard deviation of the *X* values), and vice versa.

Leoni and Costa[15] preferred to alternate (instead of varying) the charting statistic (ACS), that is, independently of the sample point position on the chart, the charting statistic always changes from one variable to the next predefined one. For instance, if the statistic of the odd samples is obtained with the *X* observations, then the statistic of the even samples is obtained with the *Y* observations.

The ACS chart is substantially easier to operate and faster than the Hotelling chart in signaling changes in the mean vector of bivariate and trivariate processes. Leoni and Costa[16] also explored the idea of monitoring bivariate and trivariate mean vectors with *np* charts applying the strategy of alternating their charting statistics. In comparison with the work of Leoni and Costa[16], we explore an alternative approach to obtain the properties of the ACS chart, that is, the Markov chain approach.

Aparisi et al[17] proposed the approach of varying the number of monitored quality characteristics with which the T^2 statistic is computed. They considered the case of p variables with a subset of p_1 variables being easy and/or inexpensive to measure and the remaining p - p_1 variables being difficult and/or expensive to measure. The current value of the charted statistic determines the next number of variables to be monitored (either p_1 or p).

In this article, we compare the average run length (ARL) of different kinds of control charts. ARL is also known as average time to signal (ATS), which means speed with which the VCS, the ACS and the T^2 charts signal changes in the mean vector of bivariate and trivariate processes. We also investigate the monitoring of four quality characteristics with the VCS T^2 chart - the statistics of the VCS chart are the two T^2 statistics obtained with the X and Y observations and with the Z and W observations.

The present manuscript is organized as follows: in section 2, we introduce the assumptions and the general notations. Sections 3, 4 and 5 deal with the monitoring of two, three, and four quality characteristics respectively. Finally, sections 6 and 7 refer to an illustrative example and the conclusions respectively.

2. Assumptions and the general notations

The monitoring of multivariate processes is the focus of this article. The main assumption is that an assignable cause changes the mean vector $\boldsymbol{\mu}$ of a multivariate normal distribution (X_1, X_2…, X_p) without affecting its covariance matrix:

$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1p}\sigma_1\sigma_p \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2p}\sigma_2\sigma_p \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{1p}\sigma_1\sigma_p & \rho_{2p}\sigma_2\sigma_p & \cdots & \sigma_p^2 \end{bmatrix}$$

During the in-control period, the mean vector $\boldsymbol{\mu}$ is $\boldsymbol{\mu}_0 = (\mu_{01},\ldots,\mu_{0p})$ and, after the occurrence of the assignable cause, the mean vector $\boldsymbol{\mu}$ is $\boldsymbol{\mu}_1 = (\mu_{11},\ldots,\mu_{1p})$. If $\boldsymbol{\delta}$ is the standardized mean shift vector, then $\boldsymbol{\delta}' = (\delta_1,\ldots,\delta_p)$, with $\delta_\Theta = (\mu_{1\Theta} - \mu_{0\Theta})/\sigma_\Theta$. The multivariate Hotelling's statistic is given by $T^2 = n(\overline{\mathbf{X}} - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu}_0)$, where $\overline{\mathbf{X}}' = (\overline{X}_1,\ldots,\overline{X}_p)$. The T^2 statistic follows a non-central chi-square distribution with the non-centrality parameter λ and p degrees of freedom (number of variables), i.e.: $T^2 \sim \chi_p^2(\lambda)$, with $\lambda = n(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0))$. In the next sections, X_1 is X, X_2 is Y, X_3 is V (trivariate case) or Z (tetravariate case), and X_4 is W.

The VCS $\overline{X}$ chart is also compared with the multivariate exponentially weighted moving average (MEWMA) chart proposed by Lowry *et al*[18]. The statistic plotted on the control chart is

$$T_i^2 = Z_i' \sum_{Z_i}^{-1} Z_i$$

where $Z_i = r(\mathbf{X}_i - \boldsymbol{\mu}_0) + (1-r)Z_{i-1}$, with $0 < r \leq 1$. The variance–covariance matrix Σ_{Z_i} of the MEWMA statistic Z_i is given by $\sum_{Z_i} = \frac{r}{2-r}\Sigma_0$.

3. Monitoring bivariate Processes

The T^2 chart is the most common chart used to control the mean vector of bivariate processes. With the T^2 chart in use, samples of size n are regularly collected and the two quality characteristics of the selected items are measured, leading to sample data set of cardinalities $2n$ (c=$2n$). Leoni and Costa[15] proposed, as an alternative to the Hotelling's chart, the Shewhart chart for the sample means combined with the following sampling-measuring strategy: at each sampling point t_i, i = {1, 2, 3 …}, $2n$ items are collected, but only 1 of the 2 quality characteristics (X, Y) is measured; the cardinality of the sample date set is also $2n$. This way, the number of observations per sample required by their chart is the same one required by the T^2 chart. If the observations of the quality characteristic X are used to obtain the charting points i = {1, 3, 5 …}, as X sample means, $(= \sum_{j=1}^{2n} X_{ij} / 2n)$, then the observations of the quality characteristic Y will be used to obtain the charting points i = {2, 4, 6 …}, as Y sample means, $(= \sum_{j=1}^{2n} Y_{ij} / 2n)$. Therefore, the chart proposed by Leoni and Costa[15] is a Shewhart chart with alternated charting statistic (ACS).

Figure 1 presents the ACS chart with four control limits $(UCL_{\overline{X}}; LCL_{\overline{X}}; UCL_{\overline{Y}}; LCL_{\overline{Y}})$:

$$UCL_{\overline{X}} = \mu_{0X} + k\sigma_X / \sqrt{2n} \qquad (1)$$

$$LCL_{\overline{X}} = \mu_{0X} - k\sigma_X / \sqrt{2n}$$ ~~(2)~~

$$UCL_{\overline{Y}} = \mu_{0Y} + k\sigma_Y / \sqrt{2n}$$ ~~(3)~~

$$LCL_{\overline{Y}} = \mu_{0Y} - k\sigma_Y / \sqrt{2n}$$ ~~(4)~~

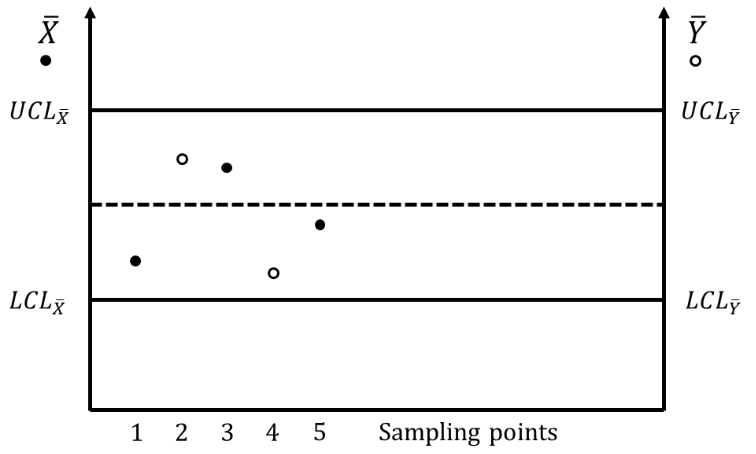

FIGURE 1: The alternated charting statistic chart

The monitoring procedure works in an alternating fashion; for instance, in Figure 1, the monitoring procedure starts by measuring the *X* quality characteristic of the first sample items, after that, the monitoring procedure switches to the second variable, by measuring the *Y* quality characteristic of the second sample items, subsequently, the monitoring procedure returns to the first variable, by measuring the *X* quality characteristic of the third sample items.

The speed with which the control chart signals is measured by the Average Run Length (*ARL*), which is the average number of samples the control chart requires to signal an out of control condition. The ~~following~~ Markov chain matrix expressed in (1) is built to obtain the *ARL*s of the ACS chart:

$$\mathrm{M} = \begin{bmatrix} 0 & p_{12} & p_{13} \\ p_{21} & 0 & p_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad (1)$$

The first and the second states of matrix M are the transient states and the third state is the absorbing one. The first state is related to the *X* observations and the second one refers to the *Y* observations. The transition probabilities are:

$$p_{12} = \Phi(k_X - \delta_X \sqrt{2n}) - \Phi(-k_X - \delta_X \sqrt{2n}) \quad (2)$$

$$p_{21} = \Phi(k_Y - \delta_Y\sqrt{2n}) - \Phi(-k_Y - \delta_Y\sqrt{2n}) \quad (3)$$

$$p_{13} = \Phi(-k_X + \delta_X\sqrt{2n}) + \Phi(-k_X - \delta_X\sqrt{2n}) \quad (4)$$

$$p_{23} = \Phi(-k_Y + \delta_Y\sqrt{2n}) + \Phi(-k_Y - \delta_Y\sqrt{2n}) \quad (5)$$

In expressions, 2, 3, 4, and 5, Φ (.) is the cumulative distribution function of the standard normal $N(0,1)$ distribution. The *ARL*s of the ACS chart are calculated by expression (6):

$$ARL = \Pi^{\mathrm{T}}\left[\text{I-Q}\right]^{-1}\mathbf{1} \quad (6)$$

In expression (6), I is the (2x2) Identity Matrix, Q is the M matrix after deleting the last column and the last row, $\Pi^{\mathrm{T}} = (1/2;1/2)$ and $\mathbf{1}^{\mathrm{T}} = (1;1)$. The initial probabilities vector $\Pi^{\mathrm{T}} = (1/2;1/2)$ means that, after the assignable cause occurrence, the Markov chain has fifty-fifty chance of starting with the first or with the second transient states. In the comparisons with the T^2 chart, the *ARL*s of the ACS chart were computed with $k = k_X = k_Y = 3.00$, that is, with an average number of samples between false alarms of 370.4 ($ARL_0 = ARL(\delta_X = \delta_Y = 0) = 370.4$).

The Variable and the Alternated Charting Statistics (VCS and ACS) strategies are slightly different from each other. When the Shewhart chart with variable charting statistic is used to control bivariate processes, samples of size 2n are regularly taken from the process, but only one of the two quality characteristics, *X* or *Y*, is measured and only one of the two statistics ($\overline{X}, \overline{Y}$) is computed. The statistic in use and the position of the current point define the statistic for the next sample. If the statistic in use is $\overline{X}$ and the sample point falls in the central region, then the statistic for the next sample changes to $\overline{Y}$. Alternatively, If the statistic in use is $\overline{X}$ and the sample point falls in the warning region, then the statistic for the next sample remains the same, that is $\overline{X}$. The same happens when $\overline{Y}$ is being measured. Figure 2 presents the VCS chart with four warning limits $(UWL_{\overline{X}}; LWL_{\overline{X}}; UWL_{\overline{Y}}; LWL_{\overline{Y}})$:

$$UWL_{\overline{X}} = \mu_{0X} + w_X\sigma_X / \sqrt{2n}$$ ~~(11)~~

$$LWL_{\overline{X}} = \mu_{0X} - w_X\sigma_X / \sqrt{2n}$$ ~~(12)~~

$$UWL_{\overline{Y}} = \mu_{0Y} + w_Y\sigma_Y / \sqrt{2n}$$ ~~(13)~~

$$LWL_{\overline{Y}} = \mu_{0Y} - w_Y \sigma_Y / \sqrt{2n} \qquad (14)$$

In the comparisons with the T^2 chart, the *ARL*s of the VCS chart were computed fixing $w = w_X = w_Y$. According to Figure 2, the monitoring procedure starts by measuring the *X* quality characteristic of the sample items. The first point is an $\overline{X}$ point and it is in the warning region, because of that, the statistic for the next sample remains the same, that is, the $\overline{X}$ statistic. The second point is a $\overline{X}$ point and it is in the central region, now the statistic for the next sample changes to the $\overline{Y}$ statistic. The third point is a $\overline{Y}$ point and it is in the warning region, because of that, the statistic for the next sample remains the same, that is, the $\overline{Y}$ statistic. The fourth point is a $\overline{Y}$ point and it is in the central region, now the statistic for the next sample changes to the $\overline{X}$ statistic. Independently of the charting statistic in use, the VCS chart signals when a sample point falls in the action region.

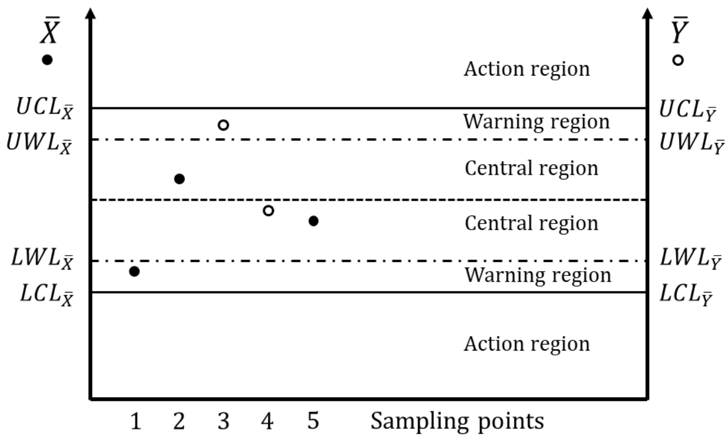

FIGURE 2: The variable charting statistic chart

The ~~following~~ Markov chain matrix M_1 expressed in (7) is built to obtain the *ARL*s of the VCS chart:

$$M_1 = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ 0 & 0 & 1 \end{bmatrix} \qquad (7)$$

The first and the second states of matrix M_1 are the transient states, and the third state is the absorbing one. The first state is related to the *X* observations and the second one is related to the *Y* observations. The transition probabilities are:

$$q_{12} = \Phi(w - \delta_X \sqrt{2n}) - \Phi(-w - \delta_X \sqrt{2n}) \quad (16)$$

$$q_{13} = \Phi(-k + \delta_X \sqrt{2n}) + \Phi(-k - \delta_X \sqrt{2n}) \quad (17)$$

$$q_{11} = 1 - q_{12} - q_{13} \quad (18)$$

$$q_{21} = \Phi(w - \delta_Y \sqrt{2n}) - \Phi(-w - \delta_Y \sqrt{2n}) \quad (19)$$

$$q_{23} = \Phi(-k + \delta_Y \sqrt{2n}) + \Phi(-k - \delta_Y \sqrt{2n}) \quad (20)$$

$$q_{22} = 1 - q_{21} - q_{23} \quad (21)$$

The *ARL*s of the VCS chart are given by Equation (8):

$$ARL = \Pi^{\mathrm{T}} \left[\mathrm{I\text{-}Q_1} \right]^{-1} \mathbf{1} \quad (8)$$

In Equation (8), I is the (2x2) Identity Matrix, $\mathrm{Q_1}$ is the $\mathrm{M_1}$ matrix after deleting the last column and the last row, $\Pi^{\mathrm{T}} = (1/2;1/2)$ and $\mathbf{1}^{\mathrm{T}} = (1;1)$.

In Table 1, the VCS and ACS $\overline{X}$ chart charts with samples of size 2 (*p*=1, *n*=2, *c*=*np*=2) are compared with the T^2 chart with samples of size 1 (*p*=2, *n*=1, *c*=*np*=2), in Table 2 the VCS and ACS $\overline{X}$ chart charts with samples of size 4 (*p*=1, *n*=4, *c*=*np*=4) are compared with the T^2 chart with samples of size 2 (*p*=2, *n*=2, *c*=*np*=4) and, in Table 3 the VCS and ACS $\overline{X}$ chart charts with samples of size 6 (*p*=1, *n*=6, *c*=*np*=6) are compared with the T^2 chart with samples of size 3 (*p*=2, *n*=3, *c*=*np*=6). The VCS $\overline{X}$ chart signals faster than the ACS $\overline{X}$ chart, except when the magnitude of the means' shifts is equal ($\delta_X = \delta_Y$), in these cases, the delays with which the VCS and the ACS $\overline{X}$ charts signal are the same. The VCS $\overline{X}$ chart always defeats the T^2 chart, except when the variables are highly correlated (ρ=0.7) and only one of them is affected by the assignable cause. In Tables 1, 2 and 3, the overall performance of the ACS, VCS and T^2 charts are measured by the expected *ARL* (*EARL*), that is, by the mean of the *ARL*s presented in these Tables for each control chart - excluding the case where $\delta_X = \delta_Y = 0$. The *EARL*s show the superiority of the ACS and VCS charts over the T^2 chart.

The ACS and VCS schemes are especially useful when the quality characteristics are evaluated by different equipment. For instance, the first one demands simple measurements, but the second one requires destructive testing.

Table 1 – Comparing bivariate charts with cardinality two

			ACS $\bar{X}$	VCS $\bar{X}$	T^2		
		ρ	-	-	0.3	0.5	0.7
		Sample size	2	2	1	1	1
		k	3	3	11.289	11.289	11.289
δ_X	δ_Y	w	-	2			
0.00	0.00		370.4	370.4	370.4	370.4	370.4
0.00	0.25		279.1	278.6	306.1	294.8	267.5*
0.00	0.50		145.5	143.2	192.5	172.2	131.8*
0.00	1.00		33.4	30.5	60.5	47.9	28.5*
0.00	1.50		9.9	8.2	20.3	15.0	8.0*
0.00	2.00		4.1	3.3	8.1	5.8	3.2*
0.25	0.25		223.9	223.9	285.4	294.8	302.3
0.25	0.50		129.0	127.9	197.8	202.2	194.5
0.25	1.00		32.5	29.9	67.1	61.5	45.1
0.25	1.50		9.8	8.2	22.7	18.9	11.7
0.25	2.00		4.1	3.3	8.9	7.1	4.1
0.50	0.50		90.6	90.6	157.1	172.2	185.4
0.50	1.00		29.4	27.9	64.1	67.3	61.8
0.50	1.50		9.6	8.2	23.3	22.1	16.3
0.50	2.00		4.1	3.4	9.4	8.2	5.4
1.00	1.00		17.7	17.7	39.8	47.9	55.8
1.00	1.50		8.0	7.5	18.9	22.1	23.3
1.00	2.00		3.8	3.4	8.8	9.4	8.3
1.50	1.50		5.3	5.3	11.9	15.0	18.2
1.50	2.00		3.1	3.1	6.7	8.2	9.3
2.00	2.00		2.3	2.3	4.6	5.8	7.2
EARL			52.3	51.3	75.7	74.9	69.4

* The cases where the T^2 defeats the VCS chart

Table 2 – Comparing bivariate charts with cardinality four

			ACS $\bar{X}$	VCS $\bar{X}$	T^2		
		ρ	-	-	0.3	0.5	0.7
		Sample size	4	4	2	2	2
		k	3	3	11.289	11.289	11.289
δ_X	δ_Y	w	-	2			
0.00	0.00		370.4	370.4	370.4	370.4	370.4
0.00	0.25		218.7	217.6	258.3	241.5	204.2*
0.00	0.50		78.2	75.1	120.3	101.6	68.8*
0.00	1.00		11.9	10.0	24.2	18.0	9.7*
0.00	1.50		3.5	2.8	6.6	4.8	2.6*
0.00	2.00		1.9	1.7	2.7	2.0	1.3*

0.25	0.25	155.2	155.2	228.2	241.5	252.5
0.25	0.50	68.3	66.8	125.4	129.8	122.2
0.25	1.00	11.7	10.0	27.6	24.7	16.8
0.25	1.50	3.5	2.9	7.5	6.2	3.8
0.25	2.00	1.9	1.7	2.9	2.4	1.6*
0.50	0.50	43.9	43.9	88.7	101.6	113.6
0.50	1.00	10.7	9.6	26.1	27.7	24.8
0.50	1.50	3.4	2.9	7.7	7.3	5.3
0.50	2.00	1.9	1.8	3.0	2.7	1.9
1.00	1.00	6.3	6.3	14.4	18.0	21.8
1.00	1.50	2.9	2.8	6.2	7.3	7.7
1.00	2.00	1.7	1.8	2.9	3.1	2.7
1.50	1.50	2.0	2.0	3.8	4.8	5.9
1.50	2.00	1.4	1.5	2.3	2.7	3.0
2.00	2.00	1.2	1.2	1.7	2.0	2.4
EARL		31.5	30.9	48.0	46.6	43.6

* The cases where the T^2 defeats the VCS chart

Table 3 – Comparing bivariate charts with cardinality six

			ACS $\bar{X}$	VCS $\bar{X}$		T^2	
		ρ	-	-	0.3	0.5	0.7
		Sample size	6	6	3	3	3
		k	3	3	11.289	11.289	11.289
δ_X	δ_Y	w	-	2			
0.00	0.00		370.4	370.4	370.4	370.4	370.4
0.00	0.25		176.4	174.6	221.6	202.2	161.9*
0.00	0.50		48.8	45.7	82.7	67.3	42.3*
0.00	1.00		6.3	5.1	12.9	9.4	5.0*
0.00	1.50		2.2	1.9	3.4	2.6	1.6*
0.00	2.00		1.6	1.6	1.6	1.3*	1.1*
0.25	0.25		115.9	115.9	187.4	202.2	214.8
0.25	0.50		42.7	41.2	87.1	90.9	84.3
0.25	1.00		6.2	5.1	14.9	13.2	8.7
0.25	1.50		2.2	1.9	3.9	3.2	2.1
0.25	2.00		1.6	1.6	1.7	1.5*	1.1*
0.50	0.50		26.4	26.4	57.1	67.3	77.1
0.50	1.00		5.8	5.1	14.0	15.0	13.3
0.50	1.50		2.1	2.0	4.0	3.8	2.8
0.50	2.00		1.5	1.6	1.8	1.6	1.3*
1.00	1.00		3.4	3.4	7.4	9.4	11.5
1.00	1.50		1.8	1.9	3.2	3.8	4.0
1.00	2.00		1.4	1.6	1.7	1.8	1.6
1.50	1.50		1.3	1.3	2.1	2.6	3.1
1.50	2.00		1.1	1.2	1.4	1.6	1.7
2.00	2.00		1.0	1.0	1.2	1.3	1.5
EARL			22.5	22.0	35.6	35.1	32.0

* The cases where the T^2 defeats the VCS chart

One of the reviewers pointed out that even with lower cardinality the ACS and VCS charts might be superior to the T^2 chart. We can see that in Table 4 and Table 5; in Table 4, the VCS and ACS $\overline{X}$ chart charts with samples of size 3 (p=1, n=3, c=np=3) are compared with the T^2 chart with samples of size 2 (p=2, n=2, c=np=4) and, in Table 5, the VCS and ACS $\overline{X}$ chart charts with samples of size 5 (p=1, n=5, c=np=5) are compared with the T^2 chart with samples of size 3 (p=2, n=3, c=np=6).

Table 4 – Comparing bivariate charts with cardinalities three and four

			ACS $\bar{X}$	VCS $\bar{X}$	T^2		
		ρ	-	-	0.3	0.5	0.7
		Sample size	3	3	2	2	2
		Cardinality	3	3	4	4	4
		k	3	3	11.289	11.289	11.289
δ_X	δ_Y	w	-	2			
0.00	0.00		370.4	370.4	370.4	370.4	370.4
0.00	0.25		246.0	245.2	258.3	241.5	204.2*
0.00	0.50		104.0	101.2	120.3	101.6	68.8*
0.00	1.00		18.6	16.2	24.2	18.0	9.7*
0.00	1.50		5.3	4.3	6.6	4.8	2.6*
0.00	2.00		2.4	2.1	2.7	2.0	1.3*
0.25	0.25		184.2	184.2	228.2	241.5	252.5
0.25	0.50		91.2	89.8	125.4	129.8	122.2
0.25	1.00		18.1	16.0	27.6	24.7	16.8
0.25	1.50		5.3	4.3	7.5	6.2	3.8*
0.25	2.00		2.4	2.1	2.9	2.4	1.6*
0.50	0.50		60.7	60.7	88.7	101.6	113.6
0.50	1.00		16.6	15.2	26.1	27.7	24.8
0.50	1.50		5.1	4.3	7.7	7.3	5.3
0.50	2.00		2.4	2.1	3.0	2.7	1.9*
1.00	1.00		9.8	9.8	14.4	18.0	21.8
1.00	1.50		4.3	4.1	6.2	7.3	7.7
1.00	2.00		2.3	2.2	2.9	3.1	2.7
1.50	1.50		2.9	2.9	3.8	4.8	5.9
1.50	2.00		1.9	1.9	2.3	2.7	3.0
2.00	2.00		1.5	1.5	1.7	2.0	2.4
EARL			39.3	38.5	48.0	46.6	43.6

* The cases where the T^2 defeats the VCS chart

Table 5 – Comparing charts bivariate charts with cardinalities five and six

			ACS $\bar{X}$	VCS $\bar{X}$		T^2	
		ρ	-	-	0.3	0.5	0.7
		Sample size	5	5	3	3	3
		Cardinality	5	5	6	6	6
		k	3	3	11.289	11.289	11.289
δ_X	δ_Y	w	-	2			
0.00	0.00		370,4	370,4	370.4	370.4	370.4
0.00	0.25		195,8	194,3	221.6	202.2	161.9*
0.00	0.50		60,9	57,7	82.7	67.3	42.3*
0.00	1.00		8,4	6,9	12.9	9.4	5.0*
0.00	1.50		2,6	2,2	3.4	2.6	1.6*
0.00	2.00		1,6	1,6	1.6	1.3*	1.1*
0.25	0.25		133,2	133,2	187.4	202.2	214.8
0.25	0.50		53,2	51,7	87.1	90.9	84.3
0.25	1.00		8,3	6,9	14.9	13.2	8.7
0.25	1.50		2,6	2,2	3.9	3.2	2.1*
0.25	2.00		1,6	1,6	1.7	1.5	1.1*
0.50	0.50		33,4	33,4	57.1	67.3	77.1
0.50	1.00		7,6	6,7	14.0	15.0	13.3
0.50	1.50		2,6	2,3	4.0	3.8	2.8
0.50	2.00		1,6	1,7	1.8	1.6*	1.3*
1.00	1.00		4,5	4,5	7.4	9.4	11.5
1.00	1.50		2,2	2,2	3.2	3.8	4.0
1.00	2.00		1,5	1,7	1.7	1.8	1.6*
1.50	1.50		1,6	1,6	2.1	2.6	3.1
1.50	2.00		1,2	1,3	1.4	1.6	1.7
2.00	2.00		1,1	1,1	1.2	1.3	1.5
EARL			26.3	25.7	35.6	35.1	32.0

* The cases where the T^2 defeats the VCS chart

In Table 6, the VCS $\overline{X}$ chart is compared with the MEWMA chart proposed by Lowry *et al*[18]. The *ARL*s of the MEWMA chart were extracted from Zang and Chang[19]. In general, the use of previous samples information reduces the delay with which the control charts signal small shifts; the MEWMA chart is not an exception.

Table 6 – Comparing the VCS $\bar{X}$ and *MEWMA* charts

			VCS $\bar{X}$	*MEWMA*
		ρ	-	0.3
		r		0.1
		Sample size	2	1
		k	2.781	8.265
δ_X	δ_Y	w	2	
0.00	0.00		185.10	184.83
0.5	0.5		51.97	20.99
1	1		11.67	8.01
2	2		1.93	3.66
3	3		1.08	2.46

4. Monitoring trivariate Processes

Leoni and Costa (2017) also investigated the properties of the trivariate ACS $\overline{X}$ chart. Similarly to the bivariate case, at each sampling point t_i, $i = \{1, 2, 3 \ldots\}$, $3n$ items are collected, but only 1 of the 3 quality characteristics (X, Y, V) is measured. This way, the number of observations per sample required by the ACS $\overline{X}$ chart is the same number required by the T^2 chart. If the observations of the quality characteristic X are used to obtain the charting points $i = \{1, 4, 7 \ldots\}$, as X sample means, then the observations of the quality characteristic Y will be used to obtain the charting points $i = \{2, 5, 8 \ldots\}$, as Y sample means. Additionally, the observations of the quality characteristic V will be used to obtain the charting points $i = \{3, 6, 9 \ldots\}$, as V sample means.

Figure 3 presents the trivariate ACS $\overline{X}$ chart with the two action limits. Now the standardized values of the sample means $Z_{\Theta} = \sqrt{3n}(\overline{\Theta} - \mu_{0\Theta}) / \sigma_{\Theta}, \Theta = \{X, Y, V\}$ are used to plot the points.

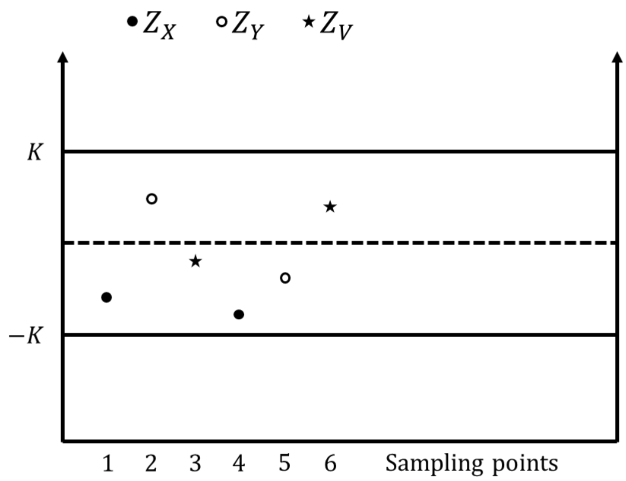

FIGURE 3: The trivariate ACS chart

The ACS chart's performance is not affected by (*XY*, *XV*, *YV*) correlations, but they affect the performance of the T^2 chart. In Tables 3 and 4, we have $\rho_{XY} = \rho_{XV} = \rho_{YV} = \rho$.

The Markov chain matrix M_2 expressed in (8) is built to obtain the *ARL*s of the trivariate ACS chart:

$$\mathrm{M}_2 = \begin{bmatrix} 0 & p_{12} & 0 & p_{14} \\ 0 & 0 & p_{23} & p_{24} \\ p_{31} & 0 & 0 & p_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The first, the second and the third states of matrix M_2 are the transient states, and the fourth state is the absorbing one. The first state is related to the *X* observations, the second state is related to the *Y* observations, and the third one is related to the *V* observations. The transient probabilities are:

$$p_{12} = \Phi(k - \delta_X \sqrt{3n}) - \Phi(-k - \delta_X \sqrt{3n})$$ ~~(24)~~

$$p_{23} = \Phi(k - \delta_Y \sqrt{3n}) - \Phi(-k - \delta_Y \sqrt{3n})$$ ~~(25)~~

$$p_{31} = \Phi(-k + \delta_V \sqrt{3n}) + \Phi(-k - \delta_V \sqrt{3n})$$ ~~(26)~~

$$p_{14} = \Phi(-k + \delta_X \sqrt{3n}) + \Phi(-k - \delta_X \sqrt{3n})$$ ~~(27)~~

$$p_{24} = \Phi(-k + \delta_Y \sqrt{3n}) + \Phi(-k - \delta_Y \sqrt{3n})$$ ~~(28)~~

$$p_{34} = \Phi(-k + \delta_V \sqrt{3n}) + \Phi(-k - \delta_V \sqrt{3n})$$ ~~(29)~~

The *ARL*s of the trivariate ACS chart are given by expression (9):

$$ARL = \Pi^{T}\left[\text{I-Q}_2\right]^{-1}\mathbf{1} \quad (9)$$

In expression (9), I is the (3X3) Identity Matrix, Q_2 is the M_2 matrix after deleting the last column and the last row, $\Pi^{T} = (1/3;1/3;1/3)$ and $\mathbf{1}^{T} = (1;1;1)$. In the comparisons with the T^2 chart, the *ARL*s of the ACS chart were computed with $k_X = k_Y = k_V = 3.00$, that is, with an average number of samples between false alarms of 370.4 ($ARL_0 = ARL(\delta_X = \delta_Y = \delta_V = 0) = 370.4$).

The Variable and the Alternated Charting Statistics (VCS and ACS) strategies are slightly different from each other. When the Shewhart chart with variable charting statistic is used to control trivariate processes, samples of size 3n are regularly taken from the process, but only one of the three quality characteristics, *X* or *Y* or *V*, is measured and only one of the three statistics ($\bar{X}, \bar{Y}, \bar{V}$) is computed. The statistic in use and the position of the current point define the statistic for the next sample. If the statistic in use is $\bar{X}$ and the sample point falls in the central region, then the statistic for the next sample changes to the $\bar{Y}$ statistic. However, if the statistic in use is $\bar{X}$ and the sample point falls in the warning region, the statistic remains the same ($\bar{X}$). Similarly, if the statistic in use is $\bar{Y}$ and the sample point falls in the central region (warning region), then the statistic for the next sample changes to the $\bar{V}$ statistic (remains the same, that is $\bar{Y}$). If the statistic in use is $\bar{V}$ and the sample point falls in the central region (warning region), then the statistic for the next sample changes to the $\bar{X}$ statistic (remains the same, that is $\bar{V}$).

According to Figure 4, the monitoring procedure starts by measuring the *X* quality characteristic of the sample items. The first point is a Z_X point and it is in the warning region, therefore the statistic for the next sample remains the same, that is, the Z_X statistic. The second point is a Z_X point and it is in the central region, now the statistic for the next sample changes to the Z_Y statistic. The third point is a Z_Y point and it is in the central region, now the statistic for the next sample changes to the Z_V statistic. The fourth point is a Z_V point and it is in the warning region, so the statistic for the next sample is still the Z_V statistic. Regardless of the charting statistic in use, the VCS chart signals when a sample point falls in the action region.

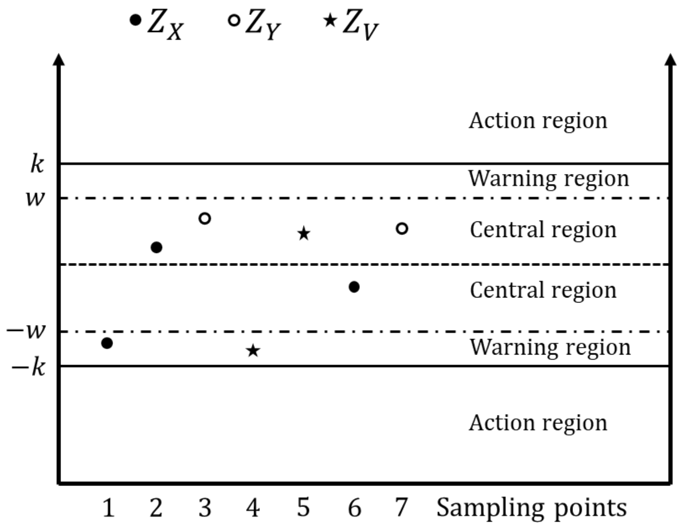

FIGURE 4: The trivariate VCS chart

The ~~following~~ Markov chain matrix M_3 expressed in (9) is built to obtain the *ARL*s of the trivariate VCS chart:

$$\mathrm{M}_3 = \begin{bmatrix} q_{11} & q_{12} & 0 & q_{14} \\ 0 & q_{22} & q_{23} & q_{24} \\ q_{31} & 0 & q_{32} & q_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

The first, the second, and the third states of matrix M_3 are the transient states, and the fourth state is the absorbing one. The first state is related to the *X* observations; the second state is related to the *Y* observations, and the third state is related to the *V* observations. The transient probabilities are:

$$q_{14} = \Phi(-k + \delta_X \sqrt{3n}) - \Phi(-k - \delta_X \sqrt{3n})$$ ~~(32)~~

$$q_{12} = \Phi(w - \delta_X \sqrt{3n}) + \Phi(-w - \delta_X \sqrt{3n})$$ ~~(33)~~

$$q_{11} = 1 - q_{12} - q_{14}$$ ~~(34)~~

$$q_{24} = \Phi(-k + \delta_Y \sqrt{3n}) - \Phi(-k - \delta_Y \sqrt{3n})$$ ~~(35)~~

$$q_{23} = \Phi(w - \delta_Y \sqrt{3n}) + \Phi(-w - \delta_Y \sqrt{3n})$$ ~~(36)~~

$$q_{22} = 1 - q_{23} - q_{24}$$ ~~(36)~~

$$q_{34} = \Phi(-k + \delta_V \sqrt{3n}) - \Phi(-k - \delta_V \sqrt{3n})$$ ~~(37)~~

$$q_{31} = \Phi(w - \delta_V \sqrt{3n}) + \Phi(-w - \delta_V \sqrt{3n})$$ ~~(38)~~

$$q_{33} = 1 - q_{31} - q_{34}$$ ~~(39)~~

The *ARL*s of the trivariate VCS chart are given by expression (10):

$$ARL = \Pi^{\mathrm{T}} \left[\mathrm{I}\text{-}\mathrm{Q}_3 \right]^{-1} \mathbf{1} \tag{10}$$

In expression (9), **I** is the (3X3) Identity Matrix, Q_3 is the M_3 matrix after deleting the last column and the last row, $\Pi^{\mathrm{T}} = (1/3;1/3;1/3)$ and $\mathbf{1}^{\mathrm{T}} = (1;1;1)$. The *ARL*s of the VCS chart were computed with $k_X = k_Y = k_V$ =3.00, that is, with an average number of samples between false alarms of 370.4 ($ARL_0 =$ $ARL(\delta_X = \delta_Y = \delta_V = 0) = 370.4$).

In Table 7 the VCS and ACS $\overline{X}$ chart charts with samples of size 3 (*p*=1, *n*=3, *c*=*np*=3) are compared with the T^2 chart with samples of size 1 (*p*=3, *n*=1, *c*=*np*=3) and, in Table 8 the VCS and ACS $\overline{X}$ chart charts with samples of size 6 (*p*=1, *n*=6, *c*=*np*=6) are compared with the T^2 chart with samples of size 2 (*p*=3, *n*=2, *c*=*np*=6). In Table 7 and 8, $\rho_{xy} = \rho_{yz} = \rho_{xz} = \rho$, but in Tables 9 and 10 they are not the same.

Table 7 – Comparing trivariate charts with cardinality three

				ACS $\overline{X}$	VCS $\overline{X}$		T^2	
			ρ	-	-	0.3	0.5	0.7
			Sample size	3	3	1	1	1
			k	3	3	14.154	14.154	14.154
δ_X	δ_Y	δ_V	w	-	2	-	-	-
0.0	0.0	0.0		370.4	370.4	370.0	370.0	370.0
0.0	0.0	0.5		136.7	132.3	213.9	187.2	138.2
0.0	0.0	1.0		27.0	22.4	73.0	53.7	28.7
0.0	0.0	2.0		3.4	2.7	9.5	6.1	2.9
0.0	0.5	0.5		84.0	82.4	168.4	156.1	121.1
0.0	0.5	1.0		24.1	20.9	71.4	59.9	37.1
0.0	0.5	2.0		3.4	2.8	10.5	7.4	3.7
0.0	1.0	1.0		14.2	13.0	42.8	36.6	22.4
0.0	1.0	2.0		3.2	2.8	9.2	7.1	3.8
0.0	2.0	2.0		2.0	1.9	4.5	3.8	2.3
0.5	0.5	0.5		60.7	60.7	163.1	187.2	206.4
0.5	0.5	1.0		21.8	19.7	79.7	85.8	76.6
0.5	0.5	2.0		3.3	2.8	12.2	10.1	5.7
0.5	1.0	1.0		14.4	12.7	52.0	59.9	57.4
0.5	1.0	2.0		3.1	2.8	11.2	10.6	6.9
0.5	2.0	2.0		2.0	1.9	5.5	5.5	3.9
1.0	1.0	1.0		9.8	9.8	40.0	53.7	67.1
1.0	1.0	2.0		2.9	2.8	10.9	12.3	10.3
1.0	2.0	2.0		1.9	1.9	5.8	7.1	6.7
2.0	2.0	2.0		1.5	1.5	4.2	6.1	8.4
	EARL			22.0	20.9	52.0	50.3	42.6

Table 8 – Comparing trivariate charts with cardinality six

				ACS $\overline{X}$	VCS $\overline{X}$		T^2	
			ρ	-	-	0.3	0.5	0.7
			Sample size	6	6	2	2	2
			k	3	3	14.154	14.154	14.154
δ_X	δ_Y	δ_V	w	-	2	-	-	-
0.0	0.0	0.0		370.4	370.4	370.0	370.0	370.0
0.0	0.0	0.5		68.6	62.9	140.0	113.2	71.6
0.0	0.0	1.0		9.2	6.8	29.4	19.8	9.3
0.0	0.0	2.0		2.1	2.1	3.0	2.0*	1.3*
0.0	0.5	0.5		38.0	36.3	96.2	85.8	59.1
0.0	0.5	1.0		8.4	6.7	28.6	22.7	12.5
0.0	0.5	2.0		2.1	2.2	3.2	2.4	1.4*
0.0	1.0	1.0		4.9	4.3	14.9	12.3	7.0
0.0	1.0	2.0		1.9	2.1	2.9	2.3	1.5*
0.0	2.0	2.0		1.4	1.4	1.6	1.4	1.1*
0.5	0.5	0.5		26.4	26.4	91.6	113.2	132.1
0.5	0.5	1.0		7.7	6.5	33.1	36.6	31.4
0.5	0.5	2.0		2.0	2.2	3.7	3.1	1.9*
0.5	1.0	1.0		4.7	4.3	19.0	22.7	21.5
0.5	1.0	2.0		2.1	2.2	3.4	3.2	2.2
0.5	2.0	2.0		1.4	1.4	1.9	1.9	1.5
1.0	1.0	1.0		3.4	3.4	13.7	19.8	26.3
1.0	1.0	2.0		1.7	2.0	3.4	3.8	3.2
1.0	2.0	2.0		1.3	1.4	2.0	2.3	2.2
2.0	2.0	2.0		1.0	1.0	1.5	2.0	2.6
	EARL			9.88	9.24	26.0	24.8	20.5

* The cases where the T^2 defeats the VCS chart

Table 9– Comparing trivariate charts with cardinality three and different correlations

				ACS $\overline{X}$	VCS $\overline{X}$		T^2	
			ρ_{xy}	-	-	0.3	0.3	0.5
			ρ_{yz}	-	-	0.5	0.3	0.5
			ρ_{xz}	-	-	0.7	0.5	0.7
			Sample size	3	3	1	1	1
			k	3	3	14.154	14.154	14.154
δ_X	δ_Y	δ_V	w	-	2	-	-	-
0.0	0.0	0.0		370.4	370.4	370.0	370.0	370.0
0.0	0.0	0.5		136.7	132.3	136.7	196.1	151.6
0.0	0.0	1.0		27.0	22.4	28.1	59.7	34.5
0.0	0.0	2.0		3.4	2.7	2.8	7.1	3.5
0.0	0.5	0.5		84.0	82.4	189.1	186.1	175.8
0.0	0.5	1.0		24.1	20.9	56.5	77.3	61.0
0.0	0.5	2.0		3.4	2.8	4.5	9.6	5.5
0.0	1.0	1.0		14.2	13.0	55.0	53.1	46.9
0.0	1.0	2.0		3.2	2.8	6.6	10.4	7.3
0.0	2.0	2.0		2.0	1.9	6.3	6.0	5.1
0.5	0.5	0.5		60.7	60.7	183.5	171.0	193.2
0.5	0.5	1.0		21.8	19.7	79.1	81.8	78.8
0.5	0.5	2.0		3.3	2.8	6.0	10.7	6.8
0.5	1.0	1.0		14.4	12.7	71.8	61.5	71.1
0.5	1.0	2.0		3.1	2.8	9.3	12.2	10.0
0.5	2.0	2.0		2.0	1.9	8.5	7.3	7.5
1.0	1.0	1.0		9.8	9.8	51.4	44.2	57.7
1.0	1.0	2.0		2.9	2.8	10.8	11.4	10.7
1.0	2.0	2.0		1.9	1.9	9.3	7.4	9.2
2.0	2.0	2.0		1.5	1.5	5.8	4.7	6.8
	EARL			22.0	20.9	48.5	53.6	49.6

Table 10– Comparing trivariate charts with cardinality six and different correlations

				ACS $\overline{X}$	VCS $\overline{X}$		T^2	
			ρ_{xy}	-	-	0.3	0.3	0.5
			ρ_{yz}	-	-	0.5	0.3	0.5
			ρ_{xz}	-	-	0.7	0.5	0.7
			Sample size	6	6	2	2	2
			k	3	3	14.154	14.154	14.154
δ_X	δ_Y	δ_V	w	-	2	-	-	-
0.0	0.0	0.0		370.4	370.4	370.4	370.4	370.4
0.0	0.0	0.5		136.7	132.3	70,5*	121,9	82,2*
0.0	0.0	1.0		27.0	22.4	9,0*	22,6	11,5*
0.0	0.0	2.0		3.4	2.7	1,2*	2,3*	1,4*
0.0	0.5	0.5		84.0	82.4	115,0	112,2	102,8
0.0	0.5	1.0		24.1	20.9	21,1	31,8	23,3
0.0	0.5	2.0		3.4	2.8	1,6*	3,0	1,9*
0.0	1.0	1.0		14.2	13.0	20,4	19,5	16,7
0.0	1.0	2.0		3.2	2.8	2,1*	3,2	2,3
0.0	2.0	2.0		2.0	1.9	2,1	2,0	1,8*
0.5	0.5	0.5		60.7	60.7	109,8	98,5	119,1
0.5	0.5	1.0		21.8	19.7	32,8	34,3	32,6
0.5	0.5	2.0		3.3	2.8	2,0*	3,3	2,2*
0.5	1.0	1.0		14.4	12.7	28,8	23,5	28,5
0.5	1.0	2.0		3.1	2.8	2,9	3,7	3,1
0.5	2.0	2.0		2.0	1.9	2,7	2,3	2,4
1.0	1.0	1.0		9.8	9.8	18,7	15,5	21,7
1.0	1.0	2.0		2.9	2.8	3,3	3,5	3,3
1.0	2.0	2.0		1.9	1.9	2,9	2,4	2,8
2.0	2.0	2.0		1.5	1.5	1,9	1,7	2,2
	EARL			22.0	20.9	23.6	26.7	24.3

* The cases where the T^2 defeats the VCS chart

The VCS $\overline{X}$ chart signals faster than the other two charts, except in two cases:

(a) When the magnitude of the means' shifts is equal ($\delta_X = \delta_Y = \delta_V$); in these situations, the delays with which the VCS and the ACS $\overline{X}$ charts signal are the same;

(b) When n=2 and the ARLs are equal or lower than 2.2; in these cases, the ACS chart signals faster than the VCS chart. If the variables are highly correlated (ρ=0.7), then both T^2 and the VCS present similar results.

Similar to the bivariate case, the trivariate ACS and VCS charts with lower cardinality defeat the T^2 chart. We can see that in Table 11, where the VCS and ACS $\overline{X}$ chart charts with samples of size 2 (p=1, n=2, c=np=2) are compared with the T^2 chart with samples of size 1 (p=3, n=1, c=np=3). In Table 12, the VCS and ACS $\overline{X}$ chart charts with samples of size 5 (p=1, n=5, c=np=5) are compared with the T^2 chart with samples of size 2 (p=3, n=2, c=np=6).

Table 11 – Comparing trivariate charts with cardinalities two and three

				ACS $\overline{X}$	VCS $\overline{X}$		T^2	
			ρ	-	-	0.3	0.5	0.7
			Sample size	2	2	1	1	1
			cardinality	2	2	3	3	3
			k	3	3	14.154	14.154	14.154
δ_X	δ_Y	δ_V	w	-	2	-	-	-
0.0	0.0	0.0		370.4	370.4	370.0	370.0	370.0
0.0	0.0	0.5		182.3	179.2	213.9	187.2	138.2*
0.0	0.0	1.0		47.8	42.3	73.0	53.7	28.7*
0.0	0.0	2.0		5.9	4.3	9.5	6.1	2.9*
0.0	0.5	0.5		121.0	119.7	168.4	156.1	121.1*
0.0	0.5	1.0		42.3	38.6	71.4	59.9	37.1*
0.0	0.5	2.0		5.8	4.4	10.5	7.4	3.7*
0.0	1.0	1.0		25.8	24.2	42.8	36.6	22.4*
0.0	1.0	2.0		5.4	4.3	9.2	7.1	3.8*
0.0	2.0	2.0		3.2	2.9	4.5	3.8	2.3*
0.5	0.5	0.5		90.6	90.6	163.1	187.2	206.4
0.5	0.5	1.0		37.9	35.6	79.7	85.8	76.6
0.5	0.5	2.0		5.7	4.4	12.2	10.1	5.7
0.5	1.0	1.0		24.1	23.2	52.0	59.9	57.4
0.5	1.0	2.0		5.3	4.3	11.2	10.6	6.9
0.5	2.0	2.0		3.2	2.9	5.5	5.5	3.9
1.0	1.0	1.0		17.7	17.7	40.0	53.7	67.1
1.0	1.0	2.0		5.0	4.3	10.9	12.3	10.3
1.0	2.0	2.0		3.1	2.9	5.8	7.1	6.7
2.0	2.0	2.0		2.3	2.3	4.2	6.1	8.4
	EARL			33.40	32.01	52.00	50.32	42.60

* The cases where the T^2 defeats the VCS chart

Table 12 – Comparing trivariate charts with cardinalities five and six

				ACS $\overline{X}$	VCS $\overline{X}$		T^2	
			ρ	-	-	0.3	0.5	0.7
			Sample size	5	5	2	2	2
			cardinality	5	5	6	6	6
			k	3	3	14.154	14.154	14.154
δ_X	δ_Y	δ_V	w	-	2	-	-	-
0.0	0.0	0.0		370.4	370.4	370.0	370.0	370.4
0.0	0.0	0.5		84.3	78.8	140.0	113.2	71.6*
0.0	0.0	1.0		12.2	9.2	29.4	19.8	9.3
0.0	0.0	2.0		2.2	2.1	3.0	2.0*	1.3*
0.0	0.5	0.5		47.8	46.0	96.2	85.8	59.1
0.0	0.5	1.0		11.1	9.0	28.6	22.7	12.5
0.0	0.5	2.0		2.2	2.2	3.2	2.4	1.4*
0.0	1.0	1.0		6.5	5.7	14.9	12.3	7.0
0.0	1.0	2.0		2.0	2.2	2.9	2.3	1.5*
0.0	2.0	2.0		1.4	1.4	1.6	1.4	1.1*
0.5	0.5	0.5		33.4	33.4	91.6	113.2	132.1
0.5	0.5	1.0		10.1	8.7	33.1	36.6	31.4
0.5	0.5	2.0		2.2	2.2	3.7	3.1	1.9*
0.5	1.0	1.0		6.2	5.7	19.0	22.7	21.5
0.5	1.0	2.0		2.0	2.3	3.4	3.2	2.2*
0.5	2.0	2.0		1.4	1.5	1.9	1.9	1.5
1.0	1.0	1.0		4.5	4.5	13.7	19.8	26.3
1.0	1.0	2.0		1.9	2.2	3.4	3.8	3.2
1.0	2.0	2.0		1.3	1.5	2.0	2.3	2.2
2.0	2.0	2.0		1.1	1.1	1.5	2.0	2.6
EARL				12.31	11.56	25.96	24.77	20.51

* The cases where the T^2 defeats the VCS chart

Phase I, where the distributional parameters are estimated, is not the focus of this article, however, it is noteworthy that the VCS chart doesn't require the estimation of the correlations, thanks to the fact that when one quality characteristic is measured the remaining quality characteristics are not.

5. Monitoring tetravariate Processes

We also applied the VCS strategy to control tetravariate processes. In this case, the four variables are split in two groups of two variables, that is, the charting statistics of the VCS chart are the Hoteling T^2 statistics obtained with the X and Y observations (T^2_{XY}) and with the Z and W observations (T^2_{ZW}).

In Table 9 and 10, the Standard T^2 chart with four variables (X, Y, W and Z) are compared with the VCS T^2 chart with the (T^2_{XY}; T^2_{ZW}) statistics. The bivariate Hotelling's statistic is given by T^2_{XY} $= 2n(\overline{\mathbf{X}}_1 - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1}(\overline{\mathbf{X}}_1 - \boldsymbol{\mu}_0)$, with $2n\overline{\mathbf{X}}_1' = (\sum_{i=1}^{2n} X_i; \sum_{i=1}^{2n} Y_i)$. The T^2_{XY} statistic follows a non-central chi-square distribution with the non-centrality parameter λ_1 and p degrees of freedom (number of variables), that is $T^2_{XY} \sim \chi^2_{p=2}(\lambda_1)$. The non-centrality parameter is given by:

$$\lambda_1 = 2n(\boldsymbol{\mu}_{11} - \boldsymbol{\mu}_{10})^T \boldsymbol{\Sigma}_{XY}^{-1}(\boldsymbol{\mu}_{11} - \boldsymbol{\mu}_{10})) \quad (9)$$

In expression (9), $\boldsymbol{\mu}_{10}' = (\mu_{0X}; \mu_{0Y})$, $\boldsymbol{\mu}_{11}' = (\mu_{1X}; \mu_{1Y})$, and $\boldsymbol{\Sigma}_{XY} = \{(\sigma_X^2, \rho_{XY}\sigma_X\sigma_Y); (\rho_{XY}\sigma_X\sigma_Y, \sigma_Y^2)\}$.

Similarly, $T^2_{XY} = 2n(\overline{\mathbf{X}}_2 - \boldsymbol{\mu}_{20})^T \boldsymbol{\Sigma}_{20}^{-1}(\overline{\mathbf{X}}_2 - \boldsymbol{\mu}_{20})$, with $2n\overline{\mathbf{X}}_2' = (\sum_{i=1}^{2n} Z_i; \sum_{i=1}^{2n} W_i)$. The T^2_{ZW} statistic follows a non-central chi-square distribution with the non-centrality parameter λ_2 and p degrees of freedom (number of variables), that is $T^2_{ZW} \sim \chi^2_{p=2}(\lambda_2)$. The non-centrality parameter is given by:

$$\lambda_2 = 2n(\boldsymbol{\mu}_{21} - \boldsymbol{\mu}_{20})^T \boldsymbol{\Sigma}_{ZW}^{-1}(\boldsymbol{\mu}_{21} - \boldsymbol{\mu}_{20})) \quad (10)$$

In expression (10), $\boldsymbol{\mu}_{20}' = (\mu_{0Z}; \mu_{0W})$, $\boldsymbol{\mu}_{21}' = (\mu_{1Z}; \mu_{1W})$, and $\boldsymbol{\Sigma}_{ZW} = \{(\sigma_Z^2, \rho_{ZW}\sigma_Z\sigma_W); (\rho_{ZW}\sigma_Z\sigma_W, \sigma_W^2)\}$. Figure 5 presents the VCS T^2 chart with action and warning limits, respectively CL and WL. According to Figure 5, the monitoring starts measuring the X and Y quality characteristics of the sample items. The first point is a T^2_{XY} point and it is in the warning region, so the statistic for the next sample remains the same, that is, the T^2_{XY} statistic. The second point is a T^2_{XY} point and it is in the central region, now the statistic for the next sample changes to the T^2_{ZW} statistic. The third point is a T^2_{ZW} point and it is in the central region, because of that, the statistic for the next sample changes to the T^2_{XY} statistic. The fourth point is a T_{XY}^2 located in the central region, so the next sample's statistics changes to T_{ZW}^2. Regardless of the charting statistic in use, the VCS T^2 chart signals when a sample point falls in the action region.

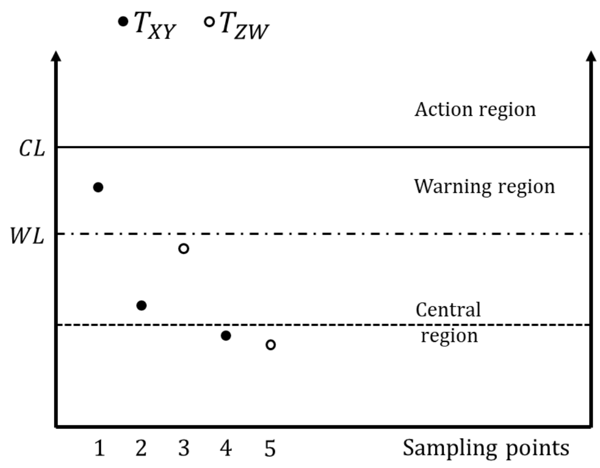

FIGURE 5: The VCS T^2 chart

The *ARL*s of the VCS T^2 chart are given by expression (5), with:

$$q_{12} = \Pr[T_{XY}^2 < WL] \tag{43}$$

$$q_{13} = \Pr[T_{XY}^2 > CL] \tag{44}$$

$$q_{11} = 1 - q_{12} - q_{13} \tag{45}$$

$$q_{21} = \Pr[T_{ZW}^2 < WL] \tag{46}$$

$$q_{23} = \Pr[T_{ZW}^2 > CL] \tag{47}$$

$$q_{22} = 1 - q_{21} - q_{23} \tag{48}$$

In Table 13, all pairs of two variables have the same correlation, that is, $\rho_{XY} = \rho_{XW} = \rho_{XZ} = \rho_{YZ} = \rho_{YW} = \rho_{ZW} = \rho$. In Table 14, the four variables are split in two groups of two variables, respecting the following condition: the pair of variables in each group are the pairs with the highest degrees of correlations. The VCS T^2 chart always defeats the standard T^2 chart, except in a few cases where all four variables are highly correlated ($\rho = 0.7$).

It is worth distinguishing between sampling unit or sample size and the cardinality of the data set. When the ACS and VCS procedures are employed, each sample unit provides a vector of observations of cardinality p=2 (p, the # of quality characteristic of interest - X and Y or Z and W), a data set after measuring $2n$ sampling units will result a matrix $2n$xp, yielding a cardinality of $4n$. When the standard T^2 chart is in use, each sample unit provides a vector of observations of cardinality p=4 (p, the # of quality characteristic of interest, X, Y, Z and W), a data set after measuring n sampling units will result a matrix nxp, yielding a cardinality of 4n. As the ACS, the VCS, and the standard T^2 charts employ the same cardinality, they are equivalent in terms of inspection effort/cost.

Table 13– Comparing the tetravariate VCS and STD T^2 charts, n=1,

$$\rho = \rho_{XY} = \rho_{XW} = \rho_{XZ} = \rho_{YZ} = \rho_{YW} = \rho_{WZ}$$

					T^2					
					VCS	STD	VCS	STD	VCS	STD
				ρ	0.3		0.5		0.7	
				Sample size	2	1	2	1	2	1
				CL	11.83	16.25	11.83	16.25	11.83	16.25
δ_X	δ_Y	δ_Z	δ_W	*WL*	2	-	2	-	2	-
0.0	0.0	0.0	0.0		370.4	370.4	370.4	370.4	370.4	370.4
0.0	0.0	0.0	1.0		36.1	83.7	26.2	60.0	13.3	30.9
0.0	0.0	0.0	2.0		3.6	11.0	2.9	6.6	2.1	2.9
0.0	0.0	1.0	1.0		20.4	46.1	26.2	34.4	32.3	17.7
0.0	0.0	1.0	2.0		3.8	9.9	4.1	6.6	3.7	3.1
0.0	0.0	2.0	0.0		3.6	11.0	2.9	6.6	2.1	2.9
0.0	0.0	2.0	2.0		2.5	4.6	2.9	3.3	3.3	1.8
0.0	1.0	1.0	1.0		17.6	37.7	18.0	34.4	12.7	21.7
0.0	1.0	1.0	2.0		4.3	10.4	4.6	8.4	4.2	4.5
0.0	1.0	2.0	2.0		3.0	5.3	3.4	4.4	3.9	2.6
0.0	2.0	2.0	2.0		2.2	3.6	2.0	3.3	1.8	2.1
1.0	1.0	1.0	1.0		14.4	41.6	18.0	60.0	21.8	77.4
1.0	1.0	1.0	2.0		4.4	13.0	4.6	15.1	4.2	12.3
1.0	1.0	2.0	2.0		3.1	7.1	3.4	8.4	3.8	6.9
1.0	2.0	2.0	2.0		2.2	5.0	2.5	6.6	2.6	6.5
2.0	2.0	2.0	2.0		1.7	4.1	2.0	6.6	2.4	9.7
	EARL				8.20	19.60	8.25	17.65	7.61	13.52

Table 14– Comparing the tetravariate VCS and STD T^2 charts, n=1,

$$\rho_{XY} = \rho_{ZW} \neq (\rho_{XZ}; \rho_{XW}; \rho_{YZ}; \rho_{YW})$$

					VCS	STD		
				$\rho_{XY}=\rho_{ZW}$		0.7		
				$\rho_{XZ}=\rho_{YZ}$	-	0.5	0.5	0.3
				$\rho_{XW}=\rho_{YW}$	-	0.5	0.3	0.3
				Sample size	2	1	1	1
				CL	11.83	16.25	16.25	16.25
δ_X	δ_Y	δ_Z	δ_W	*WL*	2	-	-	-
0.0	0.0	0.0	0.0		370.4	370.4	370.4	370.4
0.0	0.0	0.0	1.0		13.3	41.3	45.4	44.6
0.0	0.0	0.0	2.0		2.1	4.0	4.5	4.4
0.0	0.0	1.0	1.0		32.3	51.6	62.5	73.9
0.0	0.0	1.0	2.0		3.7	8.1	12.1	11.5
0.0	0.0	2.0	0.0		2.1	4.0	3.0	4.4
0.0	0.0	2.0	2.0		3.3	5.4	7.0	9.0
0.0	1.0	1.0	1.0		12.7	26.4	26.6	26.2
0.0	1.0	1.0	2.0		4.2	6.9	7.4	7.4
0.0	1.0	2.0	2.0		3.9	5.4	6.1	6.5
0.0	2.0	2.0	2.0		1.8	2.5	2.5	2.4
1.0	1.0	1.0	1.0		21.8	65.9	57.8	53.9
1.0	1.0	1.0	2.0		4.2	13.2	10.3	11.8
1.0	1.0	2.0	2.0		3.8	11.2	11.4	10.9
1.0	2.0	2.0	2.0		2.6	6.3	5.8	5.3
2.0	2.0	2.0	2.0		2.4	7.6	6.3	5.7
EARL					7.61	17.31	17.91	18.55

When the VCS scheme is applied to control four variables, we only need to estimate the X-Y and Z-W correlations, the remaining (X-Z, X-W, Y-Z, Y-W) correlations are not necessary to estimate. When the standard T^2 chart is in use monitoring four variables, we need to estimate all six correlations.

6. Illustrative Example

The example provided by Aparisi and Haro[18] is used here to show the applicability of VCS T^2 chart. Aparisi and Haro[18] presented a Figure, equal to Figure 1, to introduce the four quality characteristics of their example: the length X_1, the marked height X_2, the inner diameter X_3, and the width X_4. During the in-control period, the mean vector and the variance-covariance matrix are, respectively, $\boldsymbol{\mu}_0^{'}=(12,4,1.5,6)$ and

$$\mathbf{\Sigma}_0 = \frac{1}{10^4}\begin{bmatrix} 18 & 31 & 27 & -25 \\ 31 & 109 & -6 & 14 \\ 27 & -6 & 58 & -19 \\ -25 & 14 & -19 & 53 \end{bmatrix}$$

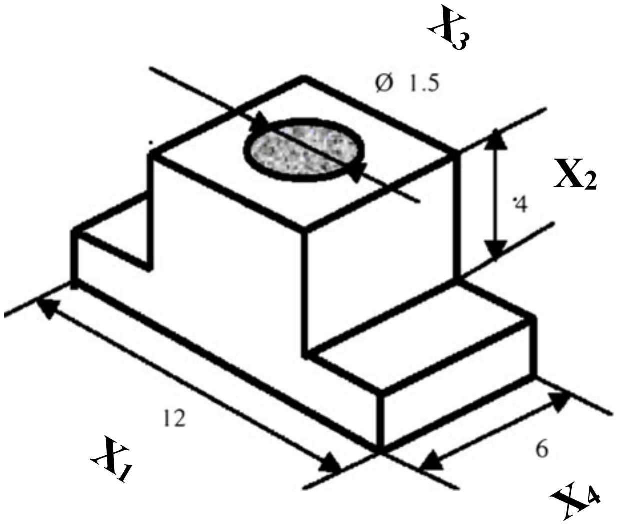

Figure 1 The four quality characteristics

When the standard T^2 chart is in use, only one item is inspected (n=1), but all four quality characteristics (X_1, X_2, X_3, X_4) are measured. If $\mathbf{X}^{\mathrm{T}}$= (X_1, X_2, X_3, X_4), then the monitoring statistics T^2 is given by $T^2 = (\mathbf{X}-\boldsymbol{\mu}_0)^{\mathrm{T}}\Sigma_0^{-1}(\mathbf{X}-\boldsymbol{\mu}_0)$. In Table 15, the first ten **X** vectors were simulated from a multivariate normal $(\boldsymbol{\mu}_0;\Sigma_0)$. The remaining five **X** vectors were simulated with the mean vector $\boldsymbol{\mu}_1^{'} = (12, 4+\sigma_{X_2}, 1.5, 6+2\sigma_{X_4})$. Table 15 also presents the values of the fifteen **($\mathbf{X}$-$\boldsymbol{\mu}_0$)** vectors and, in the last column, the T^2 values. For a fixed ARL_0=370.4, the Control Limit (CL) of the standard T^2 chart is 16.25. After sample 10, the mean vector shifted from $\boldsymbol{\mu}_0$ to $\boldsymbol{\mu}_1$; the standard T^2 chart needed five samples to signal this disturbance; in other words, sample 15 was the first sample with the T^2 value larger than the Control Limit (CL), see Figure 2.

Table 15 – The simulated observations with which the T^2 statistic was computed

#	$\mathbf{X}^T$ (X_1, X_2, X_3, X_4)				$(\mathbf{X}-\boldsymbol{\mu}_0)^T$ $(X_1\text{-}12, X_2\text{-}4, X_3\text{-}1.5, X_4\text{-}6)$				T^2
1	11.92	3.99	1.56	6.02	-0.082	-0.010	0.055	0.022	1.42
2	12.20	4.12	1.44	5.92	0.199	0.124	-0.060	-0.081	6.06
3	11.89	3.99	1.48	6.07	-0.111	-0.011	-0.022	0.072	1.37
4	12.03	3.98	1.55	5.87	0.026	-0.016	0.054	-0.129	3.30
5	12.15	4.06	1.47	6.02	0.146	0.061	-0.026	0.018	1.78
6	11.82	4.02	1.44	6.03	-0.184	0.020	-0.064	0.029	2.36
7	11.97	3.99	1.51	5.99	-0.034	-0.007	0.010	-0.011	0.14
8	12.09	4.01	1.52	6.03	0.090	0.009	0.018	0.026	0.85
9	11.92	3.85	1.51	6.03	-0.084	-0.152	0.009	0.032	2.74
10	12.21	4.07	1.52	6.05	0.209	0.069	0.018	0.049	3.68
11	11.82	4.25	1.42	6.18	-0.057	0.171	-0.005	0.173	12.17
12	11.94	4.17	1.50	6.17	0.096	0.265	-0.014	0.136	7.95
13	12.10	4.26	1.49	6.14	0.134	0.242	0.030	0.201	9.02
14	12.13	4.24	1.53	6.20	-0.016	0.062	0.102	0.243	14.18
15	11.98	4.06	1.60	6.24	-0.057	0.171	-0.005	0.173	18.15

When the VCS T^2 chart is in use, two items are inspected (n=2), but only two of the four quality characteristics (X_1, X_2, X_3, X_4) are measured, or (X_1, X_2) or (X_3, X_4). If (X_1, X_2) are measured and $(\overline{\mathbf{X}}_{12}-\boldsymbol{\mu}_{01})^{\mathrm{T}}=(\overline{X}_1-12,\overline{X}_2-4)$, then the monitoring statistics T^2 is given by:

$$T_{12}^2=(\overline{\mathbf{X}}_{12}-\boldsymbol{\mu}_{01})^{\mathrm{T}}\frac{1}{10^4}\begin{pmatrix}18 & 31\\ 31 & 109\end{pmatrix}^{-1}(\overline{\mathbf{X}}_{12}-\boldsymbol{\mu}_{01}).$$

If (X_3, X_4) are measured and $(\overline{\mathbf{X}}_{34}-\boldsymbol{\mu}_{02})^{\mathrm{T}}=(\overline{X}_3-1.5,\overline{X}_4-6)$, then the monitoring statistics T^2 is given by:

$$T_{34}^2=n(\overline{\mathbf{X}}_{34}-\boldsymbol{\mu}_{02})^{\mathrm{T}}\frac{1}{10^4}\begin{pmatrix}58 & -19\\ -19 & 53\end{pmatrix}^{-1}(\overline{\mathbf{X}}_{34}-\boldsymbol{\mu}_{02}).$$

For a fixed ARL_0=370.4, the Control Limit of the VCS T^2 chart is 11.83. The Warning Limit was fixed as 2. In Table 16, the first ten $\mathbf{X}_{ij}^{\mathrm{T}}$ (varying between $\mathbf{X}_{12}^{\mathrm{T}}$ and $\mathbf{X}_{34}^{\mathrm{T}}$) vectors were simulated from a multivariate normal $(\boldsymbol{\mu}_0;\Sigma_0)$. The last $\mathbf{X}_{34}^{\mathrm{T}}$ vector was simulated with the mean vector $\boldsymbol{\mu}_1^{'}=(12,4+\sigma_{X_2},1.5,6+2\sigma_{X_4})$. The VCS scheme works as follows: if the measurements of the current sample items are restricted to the (X_1, X_2) quality variables, then the current charting statistic is the T^2_{12} statistic. According to the T^2_{12} value, there are two possibilities for the next sample: if $T^2_{12}<WL$, then the measurements remains restricted to the (X_1, X_2) variables; otherwise, if $WL \le T^2_{12} < CL$, the measurement focus on the (X_3, X_4) quality characteristics; in this case, the charting statistic switches to the T^2_{34} statistic. The VCS scheme works in a similar way, when T^2_{34} is the current charting statistic.

In Figure 3, the first point is a T^2_{12} point; as $T^2_{12}<WL$, the second point is also a T^2_{12} point, but now $WL \le T^2_{12} < CL$, consequently, the third point is a T^2_{34} point; as $T^2_{34}<WL$, the fourth point is also a T^2_{34} point. Reminding that after sample 10, the mean vector shifted from $\boldsymbol{\mu}_0$ to $\boldsymbol{\mu}_1$; the VCS T^2 chart needed only one samples to signal this disturbance; in other words, the observations of the first sample after sample 10, lead to a T^2_{34} value larger than the Control Limit (CL), see Figure 3.

Table 16 – The simulated observations with which the T^2_{12} and T^2_{34} statistics were computed

#	$\mathbf{X}^T_{12}$ (X_1. X_2)		$\mathbf{X}^T_{34}$ (X_3. X_4)		$(\overline{\mathbf{X}}-\boldsymbol{\mu}_0)^T$ ($\overline{X}_1-12; \overline{X}_2-4$)		($\overline{X}_3-1.5; \overline{X}_4-6$)		T^2_{12}	T^2_{34}
1	11.92	3.99			-0.042	-0.047			0.50 ▲	
	12.00	3.92								
2	12.20	4.12			0.149	0.088			3.23	
	12.10	4.05								
3			1.48	6.07			0.005	0.059		1.58 ●
			1.53	6.05						
4			1.55	5.87			0.057	-0.101		4.02
			1.56	5.93						
5	12.15	4.06			0.016	-0.019			0.12	
	11.89	3.90								
6	11.82	4.02			-0.039	0.023			0.35	
	12.10	4.03								
7	11.97	3.99			-0.002	-0.020			0.08	
	12.03	3.97								
8	12.09	4.01			0.037	-0.116			3.06	
	11.98	3.76								
9			1.51	6.03			0.018	0.047		1.34
			1.53	6.06						
10			1.52	6.05			0.000	-0.014		0.08
			1.48	5.92						
11			1.42	6.18			-0.017	0.195		15.50
			1.54	6.21						

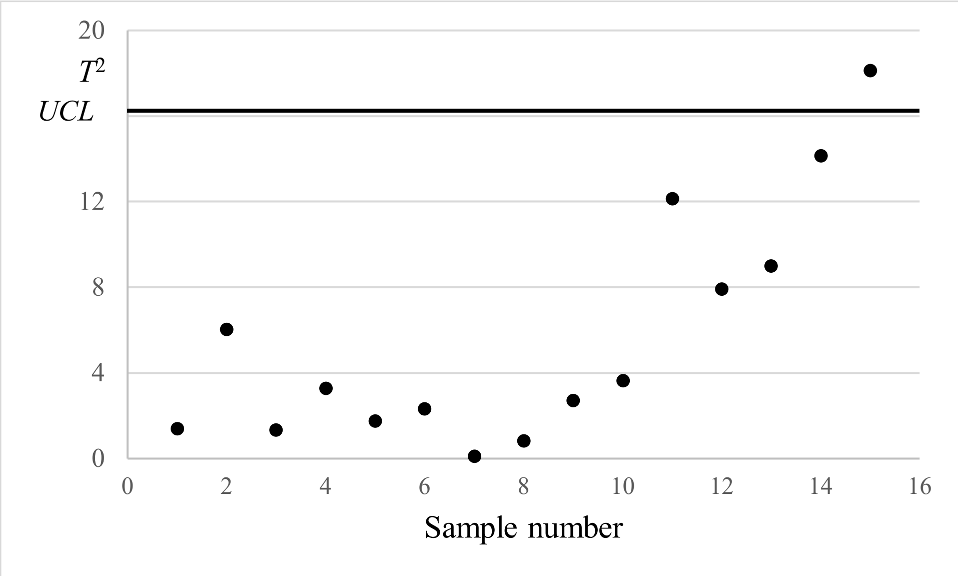

Figure 2 The standard T^2 chart

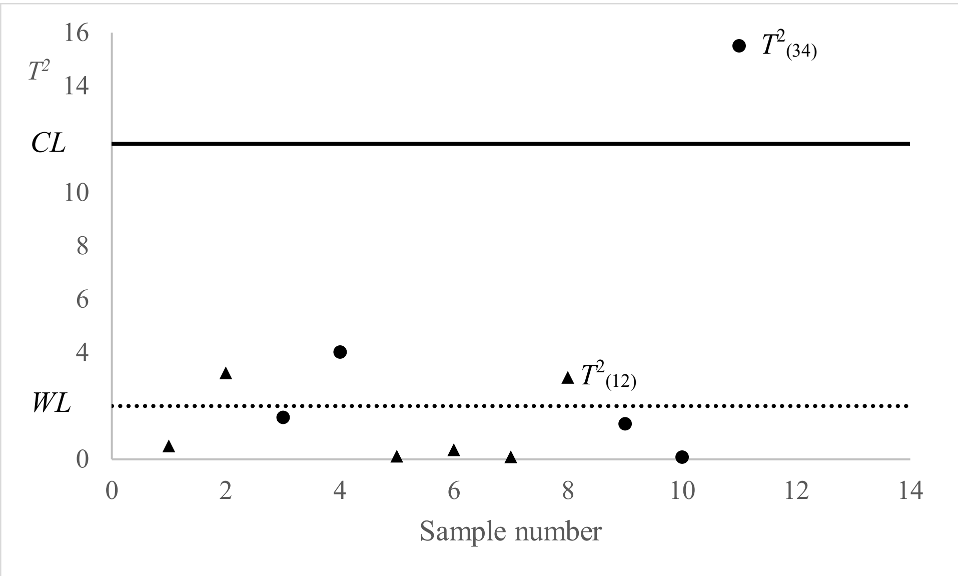

Figure 3 The VCS T^2 chart

7. Conclusions

In this article, we explored the idea of varying the control chart statistic when the number of monitored variables is larger than one. The proposed VCS chart has the two qualities the practitioner always enjoys, that is, less operational complexity and lower delays in signaling. For the monitoring of bivariate and trivariate processes, we proposed the use of the VCS $\overline{X}$ chart, and for the tetravariate processes, we proposed the use of the VCS T^2 chart.

The applicability of the VCS strategy extends to other types of control charts, such as EWMAs, CUSUMs, among many others. Future research may also include the impact of measurement errors on the performance of VCS charts, and the analysis of applications of VCS charts when data is not normally distributed.

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